

# ZEROTH DERIVED FUNCTORS AS ADJOINTS AND INJECTIVE RESOLUTIONS OF COHERENT FUNCTORS

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**ABSTRACT.** The classical definitions of zeroth derived functors require existence of injectives or projectives. In this paper, we give definitions of the zeroth derived functors that do not require the existence of injectives or projectives. The new definitions result in generalized definitions of projective and injective stabilization of functors. The category of coherent functors is shown to admit a zeroth right derived functor. An interesting result of this fact is a counterpart to the Yoneda lemma for coherent functors. Moreover, zeroth derived functors are seen more appropriately as approximations of functors by left exact or right exact functors. Under certain reasonable conditions, the category of coherent functors is shown to have enough injectives. This result was first shown by Ron Gentle. We give an alternate proof of this fact.

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## 1. THE FUNCTOR CATEGORY AND THE YONEDA LEMMA

Let  $\mathcal{C}$  and  $\mathcal{D}$  denote two abelian categories. Recall the definition of the functor category  $(\mathcal{C}, \mathcal{D})$ . The objects of  $(\mathcal{C}, \mathcal{D})$  are additive covariant functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  and the morphisms between two such objects are the natural transformations between them. The category  $(\mathcal{C}, \mathcal{D})$  is abelian. The addition of natural transformations  $\alpha, \beta$  is defined componentwise

$$(\alpha + \beta)_X := \alpha_X + \beta_X$$

A sequence of functors

$$F \rightarrow G \rightarrow H$$

is exact in  $(\mathcal{C}, \mathcal{D})$  if and only if for every  $X \in \mathcal{C}$  the sequence

$$F(X) \rightarrow G(X) \rightarrow H(X)$$

is exact in  $\mathcal{D}$ . For every  $X \in \mathcal{C}$  the evaluation functor  $\mathbf{ev}_X: (\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$  is exact.

The category of abelian groups will be denoted  $\mathbf{Ab}$ . A functor  $F \in (\mathcal{C}, \mathbf{Ab})$  is called **representable** if it is isomorphic to  $\mathbf{Hom}_{\mathcal{C}}(X, \_)$  for some  $X \in \mathcal{C}$ . The full subcategory of  $(\mathcal{C}, \mathbf{Ab})$  consisting of representable functors will be denoted  $\mathbf{Rep}(\mathcal{C}, \mathbf{Ab})$ . We will abbreviate the representable functors by  $(X, \_)$ . The most important property of representable functors is the following well known lemma of Yoneda:

**Lemma 1** (Yoneda). For any covariant functor  $F: \mathcal{C} \rightarrow \mathbf{Ab}$  and any  $X \in \mathcal{C}$

$$\mathbf{Nat}((X, \_), F) \cong F(X)$$

the isomorphism given by  $\alpha \mapsto \alpha_X(1_X)$ . This isomorphism is natural in  $F$  and  $X$ .

An immediate consequence of the Yoneda lemma is that for any  $X, Y \in \mathcal{C}$ ,  $\text{Nat}((X, \_), (Y, \_)) \cong (Y, X)$ . Hence all natural transformations between representable functors come from maps between objects in  $\mathcal{C}$ . The **Yoneda embedding** is the contravariant functor  $Y: \mathcal{C} \rightarrow (\mathcal{C}, \text{Ab})$  defined by  $Y(X) = (X, \_)$  and for any  $f: X \rightarrow Y$ ,  $Y(f) = (f, \_)$ . Notice that as a result of Yoneda's lemma, the functor  $Y$  is fully faithful. One easily shows that this embedding is also left exact. Given  $\alpha \in (Y, \_) \rightarrow (X, \_)$ , there exists  $f: X \rightarrow Y$  such that  $(f, \_) = \alpha$ . The exact sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

embeds into the exact sequence

$$0 \longrightarrow (Z, \_) \xrightarrow{(g, \_)} (Y, \_) \xrightarrow{(f, \_)} (X, \_)$$

Hence, the kernel of any natural transformation between representable functors is itself representable.

It is worth noting that the Yoneda embedding  $Y$  restricts to a functor  $\tilde{Y}: \mathcal{C} \rightarrow \text{Rep}(\mathcal{C}, \text{Ab})$  which is also dense. Therefore the category  $\mathcal{C}$  and the category  $\text{Rep}(\mathcal{C}, \text{Ab})$  are contravariantly equivalent via  $\tilde{Y}$ . As a result there exists a functor  $\tilde{w}: \text{Rep}(\mathcal{C}, \text{Ab}) \rightarrow \mathcal{C}$  such that  $\tilde{w}\tilde{Y} \cong 1$  and  $\tilde{Y}\tilde{w} \cong 1$ . Clearly  $\tilde{w}$  is defined as follows: If  $F \cong (X, \_)$ , then  $\tilde{w}(F) = X$ . Given  $(f, \_): (Y, \_) \rightarrow (X, \_)$ ,  $\tilde{w}(f, \_) = f$ . Since  $\mathcal{C}$  is abelian, so is  $\text{Rep}(\mathcal{C}, \text{Ab})$ ; however, the inclusion  $\text{Rep}(\mathcal{C}, \text{Ab}) \rightarrow (\mathcal{C}, \text{Ab})$  is not in general exact. This means that the abelian structures of  $\text{Rep}(\mathcal{C}, \text{Ab})$  and  $(\mathcal{C}, \text{Ab})$  are in general different.

Suppose that

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

is a short exact sequence in  $(\mathcal{C}, \text{Ab})$ . Consider the commutative diagram whose rows are complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Nat}((X, \_), F) & \longrightarrow & \text{Nat}((X, \_), G) & \longrightarrow & \text{Nat}((X, \_), H) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H(X) \longrightarrow 0 \end{array}$$

The vertical maps are the Yoneda isomorphisms and the bottom row is exact. This makes the top row exact. Therefore  $\text{Nat}((X, \_), \_): (\mathcal{C}, \text{Ab}) \rightarrow \text{Ab}$  is exact and as a result,  $(X, \_)$  is projective. Thus, the representable functors are projective objects in  $(\mathcal{C}, \text{Ab})$ . Throughout this paper, we will focus solely on results for covariant functors; however, in all cases, the dual results hold for contravariant functors. The easiest way to see this is to use the equivalence of the category of contravariant functors from  $\mathcal{C}$  into  $\text{Ab}$  with the category of covariant functors  $(\mathcal{C}^{op}, \text{Ab})$ . The exact details are left to the reader.

## 2. ZEROth DERIVED FUNCTORS

Throughout this section we fix abelian categories  $\mathcal{C}, \mathcal{D}$ . If  $\mathcal{C}$  has enough injectives, then given  $F \in (\mathcal{C}, \mathcal{D})$ , the classical way of defining the zeroth right derived functor  $R^0F$  is to define  $R^0F$  on its components as follows: For any  $X \in \mathcal{C}$ , take exact sequence  $0 \rightarrow X \rightarrow I^0 \rightarrow I^1$  with  $I^0, I^1$  injective. The component  $R^0F(X)$  is defined by the exact sequence

$$0 \longrightarrow R^0F(X) \longrightarrow F(I^0) \longrightarrow F(I^1)$$

It is easily seen that this assignment is functorial in both  $F$  and  $X$ . Moreover, up to isomorphism  $R^0F$  is independent of the choices of  $I^0$  and  $I^1$ . The following properties may also be established:

- (1)  $R^0F$  and  $F$  agree on injectives.
- (2)  $R^0F$  is left exact.
- (3) If  $F$  is left exact, then  $R^0F \cong F$ .
- (4) If  $G$  is left exact and  $G$  agrees on injectives with  $F$ , then  $G \cong R^0F$ .
- (5) There is a morphism  $F \rightarrow R^0F$ .
- (6) For any left exact functor  $G$ , there exists an isomorphism

$$(F, G) \cong (R^0F, G)$$

which is natural in  $F$  and  $G$

Now drop the assumption that  $\mathcal{C}$  has enough injectives. It is still possible to give a definition of zeroth right derived functors. Let  $\mathcal{S}$  denote any subcategory of  $(\mathcal{C}, \mathcal{D})$ . Define  $\text{Lex}(\mathcal{S})$  to be the full subcategory of  $\mathcal{S}$  consisting of the left exact functors, that is, all functors  $F \in \mathcal{S}$  such that if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is exact, then  $0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$  is exact. There is an inclusion functor

$$\text{Lex}(\mathcal{S}) \xrightarrow{s} \mathcal{S}$$

We say that  $\mathcal{S}$  admits a **zeroth right derived functor** if  $s$  has a left adjoint  $r^0: \mathcal{S} \rightarrow \text{Lex}(\mathcal{S})$  such that

- (1) The unit of adjunction  $u: 1_{\mathcal{S}} \rightarrow sr^0$  is an isomorphism on the injectives of  $\mathcal{C}$ . More precisely, if  $F \in \mathcal{S}$  and  $I \in \mathcal{C}$  is injective, then the morphism  $(u_F)_I$  is an isomorphism.
- (2) The composition  $r^0s$  is isomorphic to the identity. That is

$$r^0s \cong 1_{\mathcal{S}}$$

If  $\mathcal{S}$  admits a zeroth right derived functor, then we define the composition  $R^0 = sr^0$  to be the **zeroth right derived functor of  $\mathcal{S}$** . Clearly if  $\mathcal{S} \subseteq (\mathcal{C}, \mathcal{D})$  admits a zeroth right derived functor  $R^0$ , then for any functor  $F \in \mathcal{S}$  and any injective  $I \in \mathcal{C}$ ,

$$R^0F(I) \cong F(I)$$

Moreover, the functor  $r^0: \mathcal{S} \rightarrow \text{Lex}(\mathcal{S})$  produces for each functor  $F$  a left exact functor  $r^0F$  by altering  $F$  in the smallest amount possible. To clarify this comment, note that  $r^0$  does not change the functors which are already left exact. It also does not change values of  $F$  on objects  $X$  such that  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  splits for all  $Y, Z$  as this condition implies that  $0 \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z)$  is exact whether or not  $F$  is left exact. These objects are precisely the injectives of  $\mathcal{C}$ . If  $\mathcal{C}$  has enough injectives, it is easily seen that the classical definition of  $R^0$  satisfies the more general definition, that is  $(\mathcal{C}, \mathcal{D})$  admits a zeroth right derived functor.

Now suppose that  $\mathcal{S} \subseteq (\mathcal{C}, \mathcal{D})$  is an abelian subcategory of  $(\mathcal{C}, \mathcal{D})$  that admits a zeroth right derived functor  $R^0: \mathcal{S} \rightarrow \mathcal{S}$ . In this case, there is an exact sequence of functors

$$0 \longrightarrow \text{Ker}(u) \longrightarrow 1_{\mathcal{S}} \xrightarrow{u} R^0$$

where  $u$  is the unit of adjunction. The functor  $\text{Ker}(u)$  is called the **injective stabilization functor**. Given  $F \in \mathcal{S}$ , the functor  $\text{Ker}(u)(F)$  will be denoted  $\overline{F}$ . The functor  $\overline{F}$  is called the **injective stabilization of  $F$** . Evaluating the exact sequence at  $F$  yields an exact sequence of functors

$$0 \longrightarrow \overline{F} \longrightarrow F \xrightarrow{u_F} R^0F$$

A functor  $F \in \mathcal{S}$  is called **injectively stable** if  $R^0F = 0$ . This generalizes the definition of the injective stabilization given by Auslander and Bridger in [2]. The definition given there is essentially the same except that it uses the zeroth right derived functor as defined classically, which requires the existence of injectives in  $\mathcal{C}$ .

If  $\mathcal{C}$  has enough projectives, then given  $F \in (\mathcal{C}, \mathcal{D})$ , the classical way of defining the zeroth left derived functor  $L_0F$  is to define  $L_0F$  on its components as follows: For any  $X \in \mathcal{C}$  take exact sequence  $P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$  with  $P_0, P_1$  projective. The component  $L_0F(X)$  is defined by the exact sequence

$$F(P_1) \longrightarrow F(P_0) \longrightarrow L_0F(X) \longrightarrow 0$$

It is easily seen that this assignment is functorial in both  $F$  and  $X$ . Moreover, up to isomorphism  $L_0F$  is independent of the choices of  $P_0$  and  $P_1$ . The following properties may also be established:

- (1)  $L_0F$  and  $F$  agree on projectives.
- (2)  $L_0F$  is right exact.
- (3) If  $F$  is right exact, then  $L_0F \cong F$ .
- (4) If  $G$  is right exact and  $G$  agrees on projectives with  $F$ , then  $G \cong L_0F$ .
- (5) There is a morphism  $L_0F \rightarrow F$ .
- (6) For any right exact functor  $G$ , there exists an isomorphism

$$(G, F) \cong (G, L_0F)$$

which is natural in  $F$  and  $G$

Now drop the assumption that  $\mathcal{C}$  has enough projectives. It is still possible to give a definition of zeroth left derived functors. Let  $\mathcal{S}$  denote any subcategory of  $(\mathcal{C}, \mathcal{D})$ . Define  $\text{Rex}(\mathcal{S})$  to be the full subcategory of  $\mathcal{S}$  consisting of the right exact functors, that is, all functors  $F \in \mathcal{S}$  such that if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is exact, then  $F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$  is exact. There is an inclusion functor

$$\text{Rex}(\mathcal{S}) \xrightarrow{s} \mathcal{S}$$

We say that  $\mathcal{S}$  admits a **zeroth left derived functor** if  $s$  has a right adjoint  $l_0: \mathcal{S} \rightarrow \text{Rex}(\mathcal{S})$  such that

- (1) The counit of adjunction  $c: sl_0 \rightarrow 1_{\mathcal{S}}$  is an isomorphism on the projectives of  $\mathcal{C}$ . More precisely, if  $F \in \mathcal{S}$  and  $P \in \mathcal{C}$  is projective, then the morphism  $(c_F)_P$  is an isomorphism.
- (2) The composition  $l_0 s$  is isomorphic to the identity. That is

$$l_0 s \cong 1_{\mathcal{S}}$$

If  $\mathcal{S}$  admits a zeroth left derived functor, then we define the composition  $L_0 = sl_0$  to be the **zeroth left derived functor of  $\mathcal{S}$** . Clearly if  $\mathcal{S} \subseteq (\mathcal{C}, \mathcal{D})$  admits a zeroth left derived functor  $R^0$ , then for any functor  $F \in \mathcal{S}$  and any projective  $P \in \mathcal{C}$ ,

$$L_0 F(P) \cong F(P)$$

Moreover, the functor  $l_0: \mathcal{S} \rightarrow \text{Rex}(\mathcal{S})$  produces for each functor  $F$  a right exact functor  $l_0 F$  by altering  $F$  in the smallest amount possible. To clarify this comment, note that  $l_0$  does not change the functors which are already right exact. It also does not change values of  $F$  on objects  $Z$  such that  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  splits for all  $X, Y$  as this condition implies that  $F(X) \rightarrow F(Y) \rightarrow F(Z) \rightarrow 0$  is exact whether or not  $F$  is right exact. These objects are precisely the projectives of  $\mathcal{C}$ . If  $\mathcal{C}$  has enough projectives, it is easily seen that the classical definition of  $L_0$  satisfies the more general definition, that is  $(\mathcal{C}, \mathcal{D})$  admits a zeroth left derived functor.

Now suppose that  $\mathcal{S} \subseteq (\mathcal{C}, \mathcal{D})$  is an abelian subcategory of  $(\mathcal{C}, \mathcal{D})$  that admits a zeroth left derived functor  $R^0: \mathcal{S} \rightarrow \mathcal{S}$ . In this case, there is an exact sequence of functors

$$L_0 \xrightarrow{c} 1_{\mathcal{S}} \longrightarrow \text{Coker}(c) \longrightarrow 0$$

where  $c$  is the counit of adjunction. The functor  $\text{Coker}(c)$  is called the **projective stabilization functor**. Given  $F \in \mathcal{S}$ , the functor  $\text{Coker}(c)(F)$  will be denoted  $\underline{F}$ . The functor  $\underline{F}$  is called the **projective stabilization of  $F$** . Evaluating the exact sequence at  $F$  yields an exact sequence of functors

$$L_0 F \xrightarrow{c_F} F \longrightarrow \underline{F} \longrightarrow 0$$

A functor  $F \in \mathcal{S}$  is called **projectively stable** if  $L_0 F = 0$ . This generalizes the definition of the projective stabilization given by Auslander and Bridger in [2]. The definition given there is essentially the same except that it uses the zeroth left derived functor as defined classically, which requires the existence of projectives in  $\mathcal{C}$ .

### 3. COHERENT FUNCTORS

All of the results in this section are due to Auslander and can be found in [1]. Auslander defined for any abelian category  $\mathcal{C}$  the category of coherent functors  $\text{fp}(\mathcal{C}, \text{Ab})$  and studied its formal properties. In his study of  $\text{fp}(\mathcal{C}, \text{Ab})$ , he constructs an exact contravariant functor  $w: \text{fp}(\mathcal{C}, \text{Ab}) \rightarrow \mathcal{C}$  that contains certain information about  $\text{fp}(\mathcal{C}, \text{Ab})$ . Auslander's study of the functor  $w$  leads to a certain four term exact sequence of interest. We recall some important properties of  $\text{fp}(\mathcal{C}, \text{Ab})$  that will be needed later. In particular, we focus on Auslander's construction of the functor  $w: \text{fp}(\mathcal{C}, \text{Ab}) \rightarrow \mathcal{C}$  and the four term exact sequence that  $w$  induces.

A functor  $F \in (\mathcal{C}, \text{Ab})$  is **coherent** if there exists  $X, Y \in \mathcal{C}$  and exact sequence

$$(Y, \_ ) \longrightarrow (X, \_ ) \longrightarrow F \longrightarrow 0$$

The full subcategory of  $(\mathcal{C}, \text{Ab})$  consisting of coherent functors is denoted  $\text{fp}(\mathcal{C}, \text{Ab})$ . Auslander showed that  $\text{fp}(\mathcal{C}, \text{Ab})$  is abelian and the inclusion  $\text{fp}(\mathcal{C}, \text{Ab}) \rightarrow (\mathcal{C}, \text{Ab})$  is exact. The representable functors are projective objects of  $\text{fp}(\mathcal{C}, \text{Ab})$  and therefore  $\text{fp}(\mathcal{C}, \text{Ab})$  has enough projectives. In fact, the representable objects are the only projective objects of  $\text{fp}(\mathcal{C}, \text{Ab})$ .

**Proposition 2.** The only projectives of  $\mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  are the representable functors.

*Proof.* Suppose that  $F$  is a projective coherent functor. Take presentation

$$0 \longrightarrow (Z, \_ ) \longrightarrow (Y, \_ ) \longrightarrow (X, \_ ) \longrightarrow F \longrightarrow 0$$

This embeds into the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & (Z, \_ ) & \longrightarrow & (Y, \_ ) & \longrightarrow & G \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \\ 0 & \longrightarrow & (Z, \_ ) & \longrightarrow & (Y, \_ ) & \longrightarrow & (X, \_ ) \longrightarrow F \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & F \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

Since  $F$  is projective, the column  $0 \rightarrow G \rightarrow (X, \_ ) \rightarrow F \rightarrow 0$  splits and hence

$$(X, \_ ) \cong G + F$$

As a direct summand of a projective,  $G$  must be projective and hence the row  $0 \rightarrow (Z, \_ ) \rightarrow (Y, \_ ) \rightarrow G \rightarrow 0$  splits. This results in exact sequence  $0 \rightarrow G \rightarrow (Y, \_ ) \rightarrow (Z, \_ ) \rightarrow 0$ . As a result,  $G$  is a kernel of a natural transformation between representable functors. But this makes  $G$  itself representable. Now returning to the split exact sequence  $0 \rightarrow G \rightarrow (X, \_ ) \rightarrow F \rightarrow 0$ , there exists split exact sequence  $0 \rightarrow F \rightarrow (X, \_ ) \rightarrow G \rightarrow 0$  making  $F$  a kernel of natural transformation between representable functors. Consequently,  $F$  is representable. ■

At this point we will explicitly construct the functor  $w: \mathbf{fp}(\mathcal{C}, \mathbf{Ab}) \rightarrow \mathcal{C}$ . Given  $F \in \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  we take presentation

$$(Y, \_ ) \xrightarrow{(f, \_ )} (X, \_ ) \xrightarrow{\alpha} F \longrightarrow 0$$

The value of  $w$  at  $F$  is defined by the exact sequence

$$0 \longrightarrow w(F) \xrightarrow{k} X \xrightarrow{f} Y$$

This assignment gives rise to a contravariant additive functor  $w: \mathbf{fp}(\mathcal{C}, \mathbf{Ab}) \rightarrow \mathcal{C}$ . Auslander established the following properties concerning the functor  $w$ :

- (1) For any coherent functor  $F$ ,  $w(F)$  is independent of the chosen projective presentation.
- (2) For any coherent functor  $F$ ,  $w(F) = 0$  if and only if there exists an exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  such that  $0 \rightarrow (Z, \_ ) \rightarrow (Y, \_ ) \rightarrow (X, \_ ) \rightarrow F \rightarrow 0$  is exact.
- (3) The functor  $w$  is exact.

We now construct for each coherent functor  $F$  a four term exact sequence

$$0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} (w(F), \_ ) \longrightarrow F_1 \longrightarrow 0$$

that will play a major role in showing that  $\mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  admits a zeroth right derived functor. We start with the exact sequence

$$0 \longrightarrow (Z, \_ ) \xrightarrow{(g, \_ )} (Y, \_ ) \xrightarrow{(f, \_ )} (X, \_ ) \xrightarrow{\alpha} F \longrightarrow 0$$

to which we apply the exact functor  $w$  yielding the exact sequence

$$0 \longrightarrow w(F) \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

This exact sequence embeds in the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & w(F) & \xlongequal{\quad} & w(F) & & \\
 & & \parallel & & \downarrow k & & \\
 0 & \longrightarrow & w(F) & \xrightarrow{k} & X & \longrightarrow & Y \longrightarrow Z \longrightarrow 0 \\
 & & & & \downarrow & & \parallel & & \parallel \\
 0 & \longrightarrow & & & V & \longrightarrow & Y & \longrightarrow & Z \longrightarrow 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & & 
 \end{array}$$

Applying the Yoneda embedding to this diagram and extending to include cokernels where necessary yields the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & 0 & & & 0 & & \\
 & & & \downarrow & & & \downarrow & & \\
 0 & \longrightarrow & (Z, \_) & \longrightarrow & (Y, \_) & \longrightarrow & (V, \_) & \longrightarrow & F_0 \longrightarrow 0 \\
 & & \parallel & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (Z, \_) & \longrightarrow & (Y, \_) & \longrightarrow & (X, \_) & \xrightarrow{\alpha} & F \longrightarrow 0 \\
 & & & & & & \downarrow (k, \_) & & \downarrow \varphi \\
 & & & & & & (w(F), \_) & \xlongequal{\quad} & (w(F), \_) \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & F_1 & \xlongequal{\quad} & F_1 \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & 0 & & 0
 \end{array}$$

This yields the following exact sequence:

$$0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} (w(F), \_) \longrightarrow F_1 \longrightarrow 0$$

where  $w(F_0) = 0$  and  $w(F_1) = 0$ . This sequence is functorial in  $F$ . The map  $\varphi$  is the unique map such that  $\varphi\alpha = (k, \_)$ .

#### 4. THE COYONEDA LEMMA AND THE ZEROth RIGHT DERIVED FUNCTOR OF $\mathbf{fp}(\mathcal{C}, \mathbf{Ab})$

We begin this section by making the observation that if  $F \in \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  and  $w(F) = 0$ , then  $F$  vanishes on injectives. To see this note that since  $w(F) = 0$  there exists exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

such that the following sequence is exact

$$0 \longrightarrow (Z, \_ ) \xrightarrow{(g, \_ )} (Y, \_ ) \xrightarrow{(f, \_ )} (X, \_ ) \xrightarrow{\alpha} F \longrightarrow 0$$

By evaluating this sequence on an injective we have the following exact sequence

$$0 \longrightarrow (Z, I) \xrightarrow{(g, I)} (Y, I) \xrightarrow{(f, I)} (X, I) \xrightarrow{\alpha_I} F(I) \longrightarrow 0$$

Since  $I$  is injective,  $(f, I)$  is an epimorphism. Therefore  $\alpha_I = 0$ . Since  $\alpha_I$  is an epimorphism,  $F(I) = 0$ .

We now turn to one of the major properties of the functor  $w$ . It is clear from [1] that Auslander was aware of the following theorem, though he never explicitly stated it. The proof here uses Yoneda's lemma.

**Theorem 3.** Let  $F \in \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$ . Suppose that  $G \in (\mathcal{C}, \mathbf{Ab})$  is left exact. Then there are isomorphisms

$$G(w(F)) \cong (w(F), \_ ) \cong (F, G)$$

which are natural in  $F$  and  $G$ .

**Proof.** Take exact sequence

$$(Y, \_ ) \xrightarrow{(f, \_ )} (X, \_ ) \xrightarrow{\alpha} F \longrightarrow 0$$

Applying  $w$  yields the following exact sequence

$$0 \longrightarrow w(F) \xrightarrow{k} X \xrightarrow{f} Y$$

Applying the left exact functor  $G$  yields the exact sequence:

$$0 \longrightarrow G(w(F)) \xrightarrow{G(k)} G(X) \xrightarrow{G(f)} G(Y)$$

By the Yoneda lemma, this embeds into the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G(w(F)) & \xrightarrow{G(k)} & G(X) & \xrightarrow{G(f)} & G(Y) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & ((w(F), \_ ), G) & \xrightarrow{((k, \_ ), G)} & ((X, \_ ), G) & \xrightarrow{((f, \_ ), G)} & ((Y, \_ ), G) \end{array}$$

where the vertical maps are the Yoneda isomorphisms. This commutative diagram embeds into the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G(w(F)) & \xrightarrow{G(k)} & G(X) & \xrightarrow{G(f)} & G(Y) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & ((w(F), \_ ), G) & \xrightarrow{((k, \_ ), G)} & ((X, \_ ), G) & \xrightarrow{((f, \_ ), G)} & ((Y, \_ ), G) \\ & & \downarrow \theta_{F,G} & & \downarrow 1 & & \downarrow 1 \\ 0 & \longrightarrow & (F, G) & \xrightarrow{(\alpha, G)} & ((X, \_ ), G) & \xrightarrow{((f, \_ ), G)} & ((Y, \_ ), G) \end{array}$$

It is easily seen that  $\theta_{F,G}$  is an isomorphism which is natural in both  $F$  and  $G$ . Therefore

$$G(w(F)) \cong ((w(F), \_ ), G) \cong (F, G)$$

as claimed, these isomorphisms being natural in each variable. ■

The remainder of this section is devoted to interpreting the results Auslander in terms of the more general definition of zeroth right derived functors.

**Theorem 4.** The inclusion functor  $s: \text{Lex}(\mathbf{fp}(\mathcal{C}, \mathbf{Ab})) \rightarrow \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  admits a left adjoint  $r^0: \mathbf{fp}(\mathcal{C}, \mathbf{Ab}) \rightarrow \text{Lex}(\mathbf{fp}(\mathcal{C}, \mathbf{Ab}))$ . Moreover, the unit of adjunction evaluated at coherent functor  $F$  is the map  $\varphi$  in the exact sequence

$$0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} (w(F), \_) \longrightarrow F_1 \longrightarrow 0$$

**Proof.** Define  $r^0(F) := (w(F), \_)$ . From the preceding theorem, for any  $F \in \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  and for any  $G \in \text{Lex}(\mathbf{fp}(\mathcal{C}, \mathbf{Ab}))$ , there exists an isomorphism

$$(F, s(G)) \cong (r^0 F, G)$$

Moreover, this isomorphism is natural in  $F$  and  $G$ . In the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G(w(F)) & \xrightarrow{G(k)} & G(X) & \xrightarrow{G(f)} & G(Y) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & ((w(F), \_), G) & \xrightarrow{((k, \_), G)} & ((X, \_), G) & \xrightarrow{((f, \_), G)} & ((Y, \_), G) \\ & & \downarrow \theta_{F,G} & & \downarrow 1 & & \downarrow 1 \\ 0 & \longrightarrow & (F, G) & \xrightarrow{(\alpha, G)} & ((X, \_), G) & \xrightarrow{((f, \_), G)} & ((Y, \_), G) \end{array}$$

choose  $G = (w(F), \_)$  and focus on the lower left commutative square:

$$\begin{array}{ccc} \left( (w(F), \_), (w(F), \_) \right) & \xrightarrow{((k, \_), (w(F), \_))} & \left( (X, \_), (w(F), \_) \right) \\ \downarrow \theta & & \downarrow 1 \\ \left( F, (w(F), \_) \right) & \xrightarrow{(\alpha, (w(F), \_))} & \left( (X, \_), (w(F), \_) \right) \end{array}$$

By definition, the unit of adjunction evaluated at  $F$  is  $u_F = \theta(1)$ . From this commutative square, it follows that  $u_F \alpha = (k, \_)$ . Since  $\varphi$  is the unique map such that  $\varphi \alpha = (k, \_)$ , it follows that  $\varphi = u_F$ . ■

**Proposition 5.** If  $F$  is representable, then  $r^0 F \cong F$ .

**Proof.** Suppose that  $F \cong (X, \_)$ . Then  $w(F) \cong X$  and therefore

$$r^0 F = (w(F), \_) \cong (X, \_) \cong F$$

these isomorphisms being natural. ■

It is a well known fact that if  $L: \mathcal{A} \rightarrow \mathcal{B}$  and  $R: \mathcal{B} \rightarrow \mathcal{A}$  form an adjoint pair, then the unit of adjunction  $u: 1_{\mathcal{A}} \rightarrow RL$  satisfies the following property: Given  $X \in \mathcal{A}$  and  $Y \in \mathcal{B}$ , every diagram

$$\begin{array}{ccc} X & \xrightarrow{u_X} & RL(X) \\ f \downarrow & & \\ R(Y) & & \end{array}$$

embeds into a commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{u_X} & RL(X) \\
 f \downarrow & \swarrow g & \\
 R(Y) & & 
 \end{array}$$

Because the morphism  $\varphi$  in the exact sequence

$$0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} (w(F), \_ ) \longrightarrow F_1 \longrightarrow 0$$

is precisely the unit of adjunction for the adjoint pair  $(r^0, s)$  evaluated at  $F$ , we have the following property:

**Proposition 6.** Let  $F$  be a coherent functor and

$$0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} (w(F), \_ ) \longrightarrow F_1 \longrightarrow 0$$

be the corresponding four term exact sequence. Suppose that there is a natural transformation

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F_0 & \longrightarrow & F & \xrightarrow{\varphi} & (w(F), \_ ) \longrightarrow F_1 \longrightarrow 0 \\
 & & & & \eta \downarrow & & \\
 & & & & G & & 
 \end{array}$$

and  $G$  is left exact coherent. Then there exists  $\psi: (w(F), \_ ) \rightarrow G$  such that the following diagram commutes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F_0 & \longrightarrow & F & \xrightarrow{\varphi} & (w(F), \_ ) \longrightarrow F_1 \longrightarrow 0 \\
 & & & & \eta \downarrow & \swarrow \psi & \\
 & & & & G & & 
 \end{array}$$

We now come to a complete description of the left exact coherent functors.

**Theorem 7.** For any  $F \in \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$ , the following are equivalent:

- (1)  $F$  is representable.
- (2)  $F$  is projective.
- (3)  $F$  is left exact.

**Proof.** That (1) and (2) are equivalent follows from Proposition 2 and the Yoneda lemma. (1) clearly implies (3). We will show that (3) implies (2) thus completing the proof. Suppose that  $F$  is left exact. because  $F$  is left exact, the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F_0 & \longrightarrow & F & \xrightarrow{\varphi} & (w(F), \_ ) \longrightarrow F_1 \longrightarrow 0 \\
 & & & & \parallel & & \\
 & & & & F & & 
 \end{array}$$

embeds into commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F_0 & \longrightarrow & F & \xrightarrow{\varphi} & (w(F), \_ ) \longrightarrow F_1 \longrightarrow 0 \\
 & & & & \parallel & \swarrow \psi & \\
 & & & & F & & 
 \end{array}$$

Hence  $F$  is a retract of the projective  $(w(F), \_ )$ . Therefore  $F$  is projective. ■

**Theorem 8.** The category  $\mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  admits a zeroth right derived functor  $R^0: \mathbf{fp}(\mathcal{C}, \mathbf{Ab}) \rightarrow \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  and  $R^0 \cong Yw$ . In particular the zeroth right derived functor applied to a coherent functor yields a representable functor.

**Proof.** Let  $s: \mathbf{Lex}(\mathbf{fp}(\mathcal{C}, \mathbf{Ab})) \rightarrow \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  be the natural inclusion functor. Observe that for any functor  $F$ ,  $Yw(F) = (w(F), \_)$ . We have already shown that  $s$  admits a left adjoint  $r^0: \mathbf{fp}(\mathcal{C}, \mathbf{Ab}) \rightarrow \mathbf{Lex}(\mathbf{fp}(\mathcal{C}, \mathbf{Ab}))$  sending  $F$  to  $(w(F), \_)$  and that the unit of this adjunction evaluated at  $F$  is the map  $\varphi$  in the exact sequence

$$0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} (w(F), \_) \longrightarrow F_1 \longrightarrow 0$$

Since  $w(F_0) = w(F_1) = 0$ , both  $F_0$  and  $F_1$  vanish on injectives. Therefore  $\varphi_I$  is an isomorphism whenever  $I$  is injective. Finally, suppose that  $F$  is left exact. Then  $F$  is representable and therefore  $r^0 s(F) \cong F$ . ■

**Lemma 9** (The CoYoneda Lemma). For any  $F \in \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  and any  $X \in \mathcal{C}$ ,

$$\mathbf{Nat}(F, (X, \_)) \cong (X, w(F))$$

this isomorphism being natural in  $F$  and  $X$ .

**Proof.** Since  $(X, \_)$  is left exact coherent, by Theorem 3,

$$\mathbf{Nat}(F, (X, \_)) \cong (X, \_)(w(F)) = (X, w(F))$$

this isomorphism being natural in  $F$  and  $X$ . ■

## 5. INJECTIVE RESOLUTIONS OF $\mathbf{FP}(\mathcal{C}, \mathbf{AB})$

In this section it is shown that all coherent functors have injective resolutions under the assumption that  $\mathcal{C}$  has enough projectives.

**Definition 1.** Let  $\mathcal{A}$  denote an abelian category and  $\mathcal{P}$  be a full subcategory of  $\mathcal{A}$ . Define the **coherent closure** of  $\mathcal{P}$ , denoted by  $\mathcal{P}_1$ , to be the full subcategory of  $\mathcal{A}$  consisting of all objects  $X \in \mathcal{A}$  such that there exists exact sequence  $P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$  with  $P_1, P_0 \in \mathcal{P}$ .

The following result of Auslander, which can be found in [1] will be used:

**Proposition 10** (Auslander). Let  $\mathcal{A}$  denote an abelian category and  $\mathcal{P}$  be a full subcategory of  $\mathcal{A}$  satisfying the following:

- (1)  $\mathcal{P}$  consists of projectives.
- (2)  $\mathcal{P}$  is closed under finite sums.
- (3)  $\mathcal{P}$  is closed under kernels.

Under these conditions, the coherent closure  $\mathcal{P}_1$  is an abelian category and the inclusion  $\mathcal{P}_1 \rightarrow \mathcal{A}$  is exact.

**Definition 2.** Let  $\mathcal{A}$  denote an abelian category and  $\mathcal{I}$  be a full subcategory of  $\mathcal{A}$ . Define the **coherent coclosure** of  $\mathcal{I}$ , denoted by  $\mathcal{I}^1$ , to be the full subcategory of  $\mathcal{A}$  consisting of all objects  $X \in \mathcal{A}$  such that there exists exact sequence  $0 \rightarrow X \rightarrow I^0 \rightarrow I^1$  with  $I^1, I^0 \in \mathcal{I}$ .

The dual statement of Proposition 10 holds as well:

**Proposition 11.** Let  $\mathcal{A}$  denote an abelian category and  $\mathcal{I}$  be a full subcategory of  $\mathcal{A}$  satisfying the following:

- (1)  $\mathcal{I}$  consists of injectives.
- (2)  $\mathcal{I}$  is closed under finite sums.
- (3)  $\mathcal{I}$  is closed under cokernels.

Under these conditions, the coherent coclosure  $\mathcal{I}^1$  is an abelian category and the inclusion  $\mathcal{I}^1 \rightarrow \mathcal{A}$  is exact.

**Theorem 12.** If  $\mathcal{C}$  has enough projectives, then for every  $F \in \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$ , there exists injective resolution

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow 0$$

**Proof.** Let  $\mathcal{I}$  be the full subcategory of  $\mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  consisting of all functors  $H$  with presentation

$$(Q, \_ ) \rightarrow (P, \_ ) \rightarrow H \rightarrow 0$$

where  $Q, P$  are projectives in  $\mathcal{C}$ . It is easily seen that

- (1)  $\mathcal{I}$  consists of injectives in  $\mathbf{fp}(\mathcal{C}, \mathbf{Ab})$ .
- (2)  $\mathcal{I}$  is closed under finite sums.
- (3)  $\mathcal{I}$  is closed under cokernels.

By Proposition 11, the coherent coclosure  $\mathcal{I}^1$  is abelian.

Let  $X \in \mathcal{C}$ . From presentation  $P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$ , we get exact sequence

$$0 \rightarrow (X, \_ ) \rightarrow (P_0, \_ ) \rightarrow (P_1, \_ )$$

Clearly  $(P_0, \_ ), (P_1, \_ ) \in \mathcal{I}$ . Therefore  $(X, \_ ) \in \mathcal{I}^1$  which means that  $\mathcal{I}^1$  contains the representable functors. Given  $F \in \mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  we have presentation

$$(Y, \_ ) \rightarrow (X, \_ ) \rightarrow F \rightarrow 0$$

Since  $(Y, \_ ), (X, \_ ) \in \mathcal{I}^1$  and  $\mathcal{I}^1$  is abelian,  $F \in \mathcal{I}^1$ . As a result, there exists exact sequence

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1$$

where  $I^0, I^1 \in \mathcal{I}$ . Completing this sequence to include cokernels yields the following exact sequence

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow C \rightarrow 0$$

Since  $I^0, I^1 \in \mathcal{I}$  so is  $C$  because  $\mathcal{I}$  is closed under cokernels. Therefore there exists exact sequence

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow 0$$

where  $I^0, I^1, I^2 \in \mathcal{I}$ . Since  $\mathcal{I}$  consists of injectives, we have provided the desired injective resolution of  $F$ . ■

The fact that  $\mathbf{fp}(\mathcal{C}, \mathbf{Ab})$  has enough injectives whenever  $\mathcal{C}$  has enough projectives was first shown by Ron Gentle in [3], though his approach was different.

**Corollary 13.** If  $\mathcal{C}$  has enough projectives, then every coherent functor has injective dimension at most 2.

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